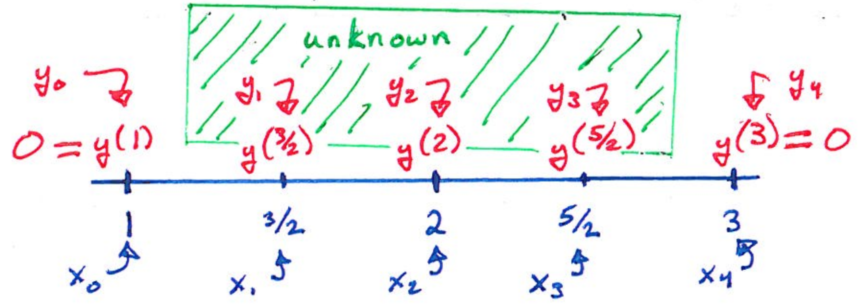
Discretize $-2y'' = 3x+1$ with $\begin{Bmatrix} y(1)=0 \\ y(3)=0 \end{Bmatrix}$ $h = 1/2$

We want to solve from $x=1$ to $x=3$ with step-size $h = 1/2$.



Convert DE to equations at each x_k, y_k

→ Recall: $y_k'' = \frac{1}{h^2}(y_{k-1} - 2y_k + y_{k+1})$

$$\cancel{-2\cdot\tfrac{1}{h^2}(y_{-1} - 2y_0 + y_1) = -2y_0'' = 3x_0 + 1 = 3\cdot 1 + 1}$$
$$-2\cdot\tfrac{1}{h^2}(y_0 - 2y_1 + y_2) = -2y_1'' = 3x_1 + 1 = 3\cdot 3/2 + 1$$
$$-2\cdot\tfrac{1}{h^2}(y_1 - 2y_2 + y_3) = -2y_2'' = 3x_2 + 1 = 3\cdot 2 + 1$$
$$-2\cdot\tfrac{1}{h^2}(y_2 - 2y_3 + y_4) = -2y_3'' = 3x_3 + 1 = 3\cdot 5/2 + 1$$
$$\cancel{-2\cdot\tfrac{1}{h^2}(y_3 - 2y_4 + y_5) = -2y_4'' = 3x_4 + 1 = 3\cdot 3 + 1}$$

(Top and bottom equations are removed because there is no y_{-1} and no y_5.)

Plug in $h = 1/2$, $\begin{Bmatrix} y_0 = 0 \\ y_4 = 0 \end{Bmatrix}$ and simplify:

$$\begin{bmatrix} -2\cdot 4(0 - 2y_1 + y_2) = 11/2 \\ -2\cdot 4(y_1 - 2y_2 + y_3) = 7 \\ -2\cdot 4(y_2 - 2y_3 + 0) = 17/2 \end{bmatrix}$$

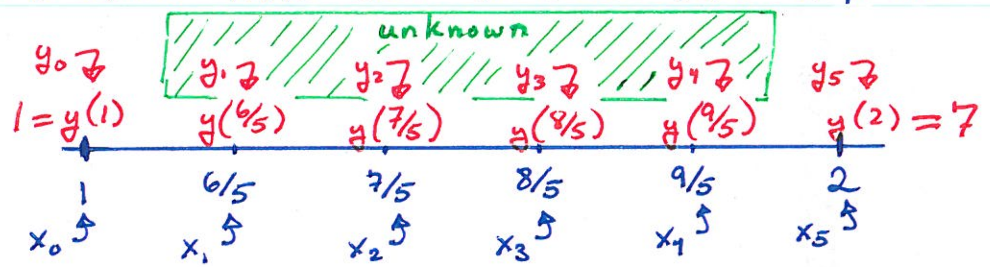
Convert to matrix equation:

$$\begin{bmatrix} 16 & -8 & 0 \\ -8 & 16 & -8 \\ 0 & -8 & 16 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 11/2 \\ 7 \\ 17/2 \end{bmatrix}$$

Note: Normally we will skip writing top & bottom equations. Also we can plug in h, y_0, y_4 immediately, instead of taking an extra step. We could even skip writing the black stuff from the middle.

Discretize $y'' + xy = 5x - 1$ w/ $\begin{Bmatrix} y(1) = 1 \\ y(2) = 7 \end{Bmatrix}$ $h = 1/5$

→ Solve from $x = 1$ to $x = 2$ with step-size $1/5$



Convert DE to equations at x_1, x_2, x_3, x_4:

$25(1 - 2y_1 + y_2) + \frac{6}{5} y_1 = \boxed{y_1'' + x_1 y_1 = 5x_1 - 1} = 5 \cdot \frac{6}{5} - 1$

$25(y_1 - 2y_2 + y_3) + \frac{7}{5} y_2 = \boxed{y_2'' + x_2 y_2 = 5x_2 - 1} = 5 \cdot \frac{7}{5} - 1$

$25(y_2 - 2y_3 + y_4) + \frac{8}{5} y_3 = \boxed{y_3'' + x_3 y_3 = 5x_3 - 1} = 5 \cdot \frac{8}{5} - 1$

$25(y_3 - 2y_4 + 7) + \frac{9}{5} y_4 = \boxed{y_4'' + x_4 y_4 = 5x_4 - 1} = 5 \cdot \frac{9}{5} - 1$

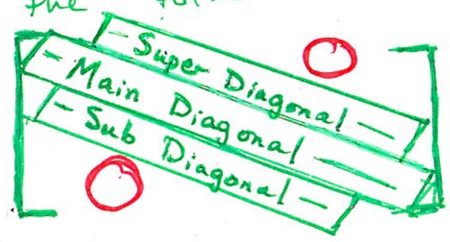
Simplify:

$$\left[\begin{array}{l} -\frac{244}{5} y_1 + 25 y_2 = 5 - 25 \\ 25 y_1 - \frac{243}{5} y_2 + 25 y_3 = 6 \\ 25 y_2 - \frac{242}{5} y_3 + 25 y_4 = 7 \\ 25 y_3 - \frac{241}{5} y_4 = 8 - 175 \end{array}\right.$$

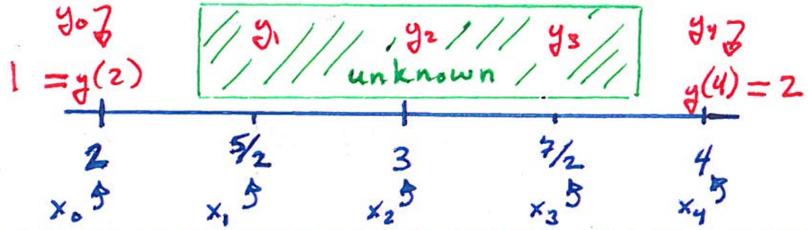
Convert to matrix equation:

$$\begin{bmatrix} -\frac{244}{5} & 25 & 0 & 0 \\ 25 & -\frac{243}{5} & 25 & 0 \\ 0 & 25 & -\frac{242}{5} & 25 \\ 0 & 0 & 25 & -\frac{241}{5} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -20 \\ 6 \\ 7 \\ -167 \end{bmatrix}$$

Note: The matrix here should always be tri-diagonal i.e. it should have the form



Discretize $-y'' + \delta(x-\tfrac{5}{2})y = 2\delta(x-3)$
with $y(2)=1$ and $y(4)=2$, $h=\tfrac{1}{2}$



Recall: $\underline{\underline{\delta}}(x_k - c) = \begin{cases} 0 & x_k \neq c \\ 1/h & x_k = c \end{cases}$ "Discrete Impulse"

Convert DE to equations at x_1, x_2, x_3:

$$\left[\begin{array}{l} -\frac{1}{h^2}(\cancelto{1}{y_0} - 2y_1 + y_2) + \cancelto{1/h}{\delta(\tfrac52 - \tfrac52)}\, y_1 = 2\cdot \cancelto{0}{\delta(\tfrac52 - 3)} \\ -\frac{1}{h^2}(y_1 - 2y_2 + y_3) + \cancelto{0}{\delta(3-\tfrac52)}\, y_2 = 2\cdot \cancelto{1/h}{\delta(3-3)} \\ -\frac{1}{h^2}(y_2 - 2y_3 + \cancelto{2}{y_4}) + \cancelto{0}{\delta(\tfrac72 - \tfrac52)}\, y_3 = 2\cdot \cancelto{0}{\delta(\tfrac72 - 3)} \end{array}\right.$$

Simplify:

$$\left[\begin{array}{l} -4(1 - 2y_1 + y_2) + 2y_1 = 0 \\ -4(y_1 - 2y_2 + y_3) = 2\cdot 2 \\ -4(y_2 - 2y_3 + 2) = 0 \end{array}\right.$$

$$\left[\begin{array}{l} 10y_1 - 4y_2 = 4 \\ -4y_1 + 8y_2 - 4y_3 = 4 \\ -4y_2 + 8y_3 = 8 \end{array}\right.$$

Matrix equation:

$$\begin{bmatrix} 10 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix}$$